Stat 455 Linear Models: Lecture 7-8 - Gaussian Mixed Model: Computation with Closed Form and EM Algorithm (09-20-2007)

1 Introduction

We introduce closed-form computation with profile log-likelihood function and EM algorithm for Gaussian mixed models.

2 Closed-Form Computation for Gaussian Mixed Model: Profile Log-likelihood

2.1 Model

For Gaussian mixed model

$$Y = X\beta + Z\alpha + \epsilon \tag{1}$$

where,

1) $\epsilon \sim \mathrm{N}(0,\tau^2 I_n)$

2) G=Var(α)=$\tau^2(UU')_{m,m}$

3) where $U = U(\psi)$ with a vector of ψ (driving profile likelihood).

2.2 Simplify Likelihood by Orthogonal Decomposition

Conditional on ψ,

1) (p 29, Textbook) The joint likelihood function $f(y, \alpha)$ is $f(y|\alpha)f(\alpha)$ = (Insert)

2) (Insert Notations)

3) After simplification, the log-likelihood function is

$$l = c - \frac{n}{2}\log(\tau^2) - \frac{1}{2}\log(|I_m + UU'Z'Z|) - \frac{1}{2\tau^2}|y^* - X^*\beta|^2 \tag{2}$$

(See attached formula file)

2.3 Estimation Procedure

1) Estimate $\hat{\beta}$ (1.46, p 30)

2) Estimate $\hat{\tau}^2$ (1.47, p 30)

3) Maximize w.r.t ψ (recall that $U = U(\psi)$, profile log-likelihood function of ψ)?

2.4 Example: One-way Random Effects Model

What is the procedure? (Textbook, p 31)

3 Expectation-Maximization (EM) Algorithm

Dempster, Laird and Rubin (1977).

3.1 EM: General Procedure

1) EM algorithm works on complete data (observed and unobserved/latent random variables).

2) Let $w = (y, \xi)$=(observed,latent).

3) Let complete data $w = (y, \xi)$ follow probability density $f(w|\theta)$.

1) **E(xpectation) Step**

$$Q(\theta|\theta^{(k)}) = E[\log f(w|\theta)|y, \theta^{(k)}] \tag{3}$$

where $\theta^{(k)}$ is the estimate at step k.

Remark: sometimes it is more clear to adopt a **Bayesian** flavor (posterior distribution for latent variables given model parameters $\theta^{(k)}$ and observed data y).

2) **M(aximization) Step**

Maximize

$$Q(\theta|\theta^{(k)}) = E[\log f(w|\theta)|y, \theta^{(k)}] \tag{4}$$

wrt θ and get updated $\theta^{(k+1)}$.

3.2 Applications to Gaussian Mixed Model

Textbook (Chapter 4.1.1). p 165. For Gaussian mixed model

$$Y = X\beta + \sum_{r=1}^{s} Z_r \alpha_r + \epsilon \tag{5}$$

1) What are (y, ξ)=(observed,latent) in above model?

2) How to prove the materials on page 165 (textbook)?

4 Appendix: Formula

5 References

- ◇ Dempster, A. P., Laird, N. M. and Rubin, D.B. (1977). Maximum likelihood from incomplete data via the EM algorithm, Journal of Royal Statistical Society B, Vol 39 No 1: 1-38.
- ◇ Jiming Jiang (2007), Linear and Generalized Linear Mixed Models and Their Applications. Springer.