[1] Likelihood function : $f(y, \alpha)$

$$f(y|\alpha)\, f(\alpha) = \frac{1}{(2\pi\tau^2)^{n/2}} \exp\left(-\frac{1}{2\tau^2}|y - X\beta - Z\alpha|^2\right) \qquad f(y|\alpha)$$

$$\times \frac{1}{(2\pi\tau^2)^{m/2}|U|} \exp\left(-\frac{1}{2\tau^2}|U^{-1}\alpha|^2\right) \qquad \boxed{f(\alpha)}$$

$$= \frac{1}{(2\pi\tau^2)^{(m+n)/2}|U|} \exp\left(-\frac{1}{2\tau^2}|\tilde{y} - \tilde{X}\beta - \tilde{Z}\alpha|^2\right)$$

where, $\tilde{y} = \begin{pmatrix} y \\ 0 \end{pmatrix}$, $\tilde{X} = \begin{pmatrix} X \\ 0 \end{pmatrix}$, $\tilde{Z} = \begin{pmatrix} Z \\ U^{-1} \end{pmatrix}$

[2] $\tilde{u} := \tilde{y} - \tilde{X}\beta$, and $\tilde{\alpha} = (\tilde{Z}'\tilde{Z})^{-1}\tilde{Z}'\tilde{u}$ so that

Projection:

① $\tilde{Z}\tilde{\alpha} = \tilde{Z}(\tilde{Z}'\tilde{Z})^{-1}\tilde{Z}'\tilde{u} = P_{\tilde{Z}}(\tilde{u})$

② $\tilde{u} - \tilde{Z}\tilde{\alpha} = y^* - X^*\beta$, where

$$y^* = P_{\tilde{Z}^\perp}\tilde{y}, \quad X^* = P_{\tilde{Z}^\perp}\tilde{X}$$

(proof?)

[1] log-likelihood for complete data

$$l = c - \frac{1}{2}\left\{ n\log(\tau^2) + \sum_{r=1}^{s} m_r \log(\sigma_r^2) + \sum_{r=1}^{s} \frac{\alpha_r'\alpha_r}{\sigma_r^2} + \frac{1}{\tau^2}\left(y - X\beta - \sum_{r=1}^{s} Z_r\alpha_r\right)'\left(y - X\beta - \sum_{r=1}^{s} Z_r\alpha_r\right) \right\}$$

[2] E-step,

$$\begin{cases} E(\alpha_r|y) = \sigma_r^2 Z_r' V^{-1}(y - X\beta) \\ E(\alpha_r'\alpha_r|y) = \sigma_r^4 (y - X\beta)' V^{-1} Z_r Z_r' V^{-1}(y - X\beta) \\ \quad + \sigma_r^2 m_r - \sigma_r^4 \mathrm{tr}(Z_r' V^{-1} Z_r), \quad 1 \leq r \leq s \end{cases}$$

where, $V = Var(y) = \tau^2 I_n + \sum_{r=1}^{s} \sigma_r^2 Z_r Z_r'$

[3] M-Step:

$$\begin{cases} (\sigma_r^2)^{(k+1)} = m_r^{-1} E(\alpha_i' \alpha_i | y) \Big|_{\beta=\beta^{(k)},\ \sigma^2=\sigma^{2(k)}}, \quad 1 \le r \le s \\ \beta^{(k+1)} = (X'X)^{-1} X' \left\{ y - \sum_{q=1}^{s} Z_q E(\alpha_q | y) \Big|_{\beta=\beta^{(k)},\ \sigma^2=(\sigma^2)^{(k)}} \right\} \end{cases}$$

where, $\sigma^2 = (\sigma_j^2)$ $1 \le j \le s$.